NOTE ON BOUNDARY LAYER FLOWS OF NON-NEWTONIAN POWELL–EYRING FLUIDS

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The boundary layer equations for flow of non-Newtonian Powell–Eyring fluids past a rectangular wedge are solved exactly over a wide range of the governing parameters. The accuracy of an approximate solution of the same flow problem, based on the integral momentum theorem, is found to be sufficient for rapid engineering calculations.

Problems of boundary layer flows of non-Newtonian fluids have been solved up to now predominantly for the power-law model of purely viscous behaviour[1–3]. For this automorphous type of viscosity function[4,5] it has been stated several times, that similarity transformations can be usually found for the same flow situations as in Newtonian fluid mechanics. In the resulting similarity equations one new parameter appears, namely the dimensionless flow index n of the power-law model used.

Analysis of analogous problems for other, non-automorphous non-Newtonian fluids are very scarce. Besides the general analysis by Lee and Ames[6], the paper by Hansen and Na[7] is exceptional.

These authors tried to find conditions for the existence of similarity transformations for steady plane two-dimensional laminar boundary layer flows of such purely viscous fluids, the relation between the shear stress and the rate of shear of which can be expressed by an arbitrary continuous function. They concluded that similarity solutions are possible only if the velocity distribution of the potential flow $U(x)$ could be expressed as $U(x) \sim x^{1/3}$, what is often interpreted[8] as a flow past a wedge of 90 deg, with constant value of the shear stress at the solid wall. Hansen and Na[7] extended their general analysis to the case of Powell–Eyring (PE) fluids, considering just a relatively narrow range of governing parameters; moreover the final presentation of their results is not very illustrative.

Therefore we decided to carry out the analysis of this case over a broader interval of parameters which occurs with real PE fluids to get better insight into the structure of this problem involving non-power-law boundary layer flows past solid obstacles. The results of the exact (similarity) solution are then used for checking the accuracy of results obtained for the same case approximatively by the method of integral momentum balance. The results obtained could serve also as basis for the extension of the pseudosimilarity concept[9-11] into the field of external boundary layer flows of non-Newtonian fluids.

THEORETICAL

EXACT SOLUTION

The problem under consideration is described by the following system of equations: continuity

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0\,, \tag{1}$$

momentum

$$\varrho\left(u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y}\right) = \frac{\partial \tau}{\partial y} + \varrho U\frac{\mathrm{d}U}{\mathrm{d}x}\,, \tag{2}$$

constitutive relation

$$\tau = \tau_{yx} = \mu\frac{\partial u}{\partial y} + \frac{1}{B}\sinh^{-1}\left(\frac{1}{C}\frac{\partial u}{\partial y}\right), \tag{3}$$

with boundary conditions

$$y = 0: \quad u = v = 0\,,$$
$$y \to \infty: \quad u \to U(x) = ex^{1/3}\,, \tag{4a,b}$$

Introducing a characteristic velocity U_0 and a characteristic length L and expressing u and v with the aid of the dimensionless velocity function $G(\xi)$ as

$$u = ex^{1/3}\,G'(\xi) \tag{5}$$

$$v = -\tfrac{1}{3}L^{1/3}U_0x^{-1/3}\left(\frac{3}{\mathrm{Re}}\right)^{1/2}(2G - G'\xi)\,, \tag{6}$$

where $\mathrm{Re} = \varrho U_0 L/\mu$, $e = U_0 L^{-1/3}$ and the primes indicate differentiation with respect to the so-called similarity variable

$$\xi = yx^{-1/3}\,L^{-2/3}\left(\frac{\mathrm{Re}}{3}\right)^{1/2}\,, \tag{7}$$

Eqs (2) and (3) are transformed into the form of a single ordinary differential (similarity) equation

$$G''' = \frac{(G'^2 - 2GG'' - 1)(\beta G''^2 + 1)^{1/2}}{(\beta G''^2 + 1)^{1/2} + \mathrm{Ey}} \tag{8}$$

with the two parameters

$$\mathrm{Ey} = 1/\mu BC \quad \text{and} \quad \beta = \frac{1}{3}\frac{\varrho U_0^3}{L}\frac{1}{C^2\mu}\,. \tag{9*}$$

In order to get the velocity profiles (*5*) and (*6*), equation (*8*) must be solved with the boundary conditions

$$\xi = 0: \quad G = G' = 0\,, \tag{10a,b}$$

$$\xi \to \infty: \quad G' \to 1\,,$$

as an initial value problem, *e.g.* by the "shooting" method: keeping Ey and β constant, $G''(0)$ must be altered so long as an asymptotic behaviour of $G'(\xi)$ in the sense of (*10b*) is achieved with a prescribed accuracy on a previously unknown, but finite interval ξ_∞. The value of $G''(0)$ obtained in this way may be used to estimate the dimensionless value T_c

$$T_c = \frac{\tau_w}{\mu(U_0/L)(\mathrm{Re}/3)^{1/2}} = G''(0) + \frac{\mathrm{Ey}}{\sqrt{\beta}}\sinh^{-1}[\sqrt{\beta}\,G''(0)]\,, \tag{11}$$

from which the shearing stress on the surface of the plate τ_w can be estimated.

Approximate Solution by the Method of Integral Momentum Balance

Integrating the equation of motion (*2*) along the thickness of the boundary layer $\delta(x)$, we get (details see in[8]) so-called integral momentum equation for two dimensional plane flows of purely viscous fluids past solid obstacles

$$\varrho\left[\frac{\mathrm{d}}{\mathrm{d}x}\int_0^\delta u(U - u)\,\mathrm{d}y + \frac{\mathrm{d}U}{\mathrm{d}x}\int_0^\delta (U - u)\,\mathrm{d}y\right] = \tau_w\,. \tag{12}$$

For PE fluids, a constitutive relation of type (*3*) must be substituted for the shear stress at the wall τ_w.

After introducing the dimensionless coordinates $X = x/L$ and $Z = y/\delta$, the di-

* In paper[7] $\alpha \equiv 1/\mathrm{Ey}$.

mensionless longitudinal velocity for flow past a wedge with 90 deg angle

$$u = ex^{1/3}\,\mathcal{G}(Z) \tag{13}$$

and the dimensionless thickness of the boundary layer

$$\xi_a = \delta x^{-1/3} L^{-2/3} \left(\frac{\text{Re}}{3}\right)^{1/2} \tag{14}$$

Eq. (*12*) takes the form

$$\int_0^1 \mathcal{G}(1 - \mathcal{G})\,\mathrm{d}Z \,.\, \frac{\mathrm{d}}{\mathrm{d}X}(X\xi_a) + \frac{\xi_a}{3}\int_0^1 (1 - \mathcal{G})\,\mathrm{d}Z = $$
$$= \frac{\mathcal{G}'(0)}{3\,\xi_a} + \frac{\text{Ey}}{3\sqrt{\beta}} \sinh^{-1}\left[\frac{\mathcal{G}'(0)}{\xi_a}\sqrt{\beta}\right]. \tag{15}$$

Denoting the integrals in (*15*) as

$$\int_0^1 (1 - \mathcal{G})\,\mathrm{d}Z = k_1\,, \quad \int_0^1 \mathcal{G}(1 - \mathcal{G})\,\mathrm{d}Z = k_2 \tag{16}$$

and taking – according to the results of the exact solution – ξ_a to be a constant, we get the final form of the integral momentum balance for the case under consideration

$$k_2\xi_a + \frac{\xi_a}{3}k_1 = \frac{\mathcal{G}'(0)}{3\xi_a} + \frac{\text{Ey}}{3\sqrt{\beta}} \sinh^{-1}\left[\frac{\mathcal{G}'(0)}{\xi_a}\sqrt{\beta}\right]. \tag{17}$$

This equation can be solved for ξ_a after assuming a suitable relationship for $\mathcal{G}(Z)$, the so-called velocity profile in the boundary layer. Then we are able to estimate – in analogy to T_e – the approximate dimensionless value T_a, which is a basic characteristics of the case under consideration, from the relation

$$T_a = \frac{\mathcal{G}'(0)}{\xi_a} + \frac{\text{Ey}}{\sqrt{\beta}} \sinh^{-1}\left[\frac{\mathcal{G}'(0)}{\xi_a}\sqrt{\beta}\right]. \tag{18}$$

The degree of accuracy of the approximate method, which is influenced mainly by the form of the velocity profile $\mathcal{G}(Z)$ chosen, may be tested by comparing T_e and T_a according to (*11*) and (*18*).

NUMERICAL CALCULATIONS

Exact Solution

The equation (*8*) was solved numerically by the Runge–Kutta's method on a Tesla 200 computer. The range of the main material characteristics of PE fluids, the so called Eyring number Ey, Ey $\in (0; 3000)$, was chosen according to values of individual material constants of PE fluids, found in the literature[12,13]. The results are presented in Fig. 1 in terms of the ratio $\Delta T_e = T_e/T_{e,\mathrm{Newt}}$, where $T_{e,\mathrm{Newt}} = 1{\cdot}3120$ (see[7,8]).

Approximate Solution

In spite of the fact that the results of the approximate solution depend to a great extent on the form of the velocity profile $\mathscr{G}(Z)$, there exist no unambiguous recommendations for a rational choice of such a relation. In solutions of similar problems the form of the velocity profile chosen often differs from case to case.

For all our calculations the simple velocity profile

$$\mathscr{G}(Z) = 1 - (1 - Z)^3 \tag{19}$$

has been used, mainly because of these reasons: *a*) Relation (*19*) fulfills the most important boundary conditions of the problem, which arise from the physical substance of the process, $\mathscr{G}(0) = 0$ and $\mathscr{G}(1) = 1$, and also the condition of compatibility at the outer edge of the boundary layer $\mathscr{G}'(1) = 0$. *b*) In preliminary calculations[14]

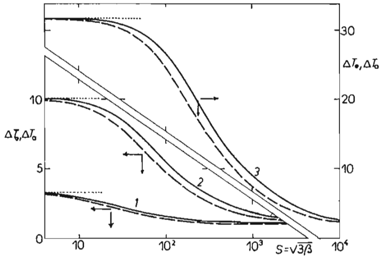

FIG. 1

Ratios of ΔT_e (solid lines) and ΔT_a (broken lines) in Dependence on Dimensionless Shear Rate S

1 — Ey = 10, *2* — Ey = 100, *3* — Ey = 1000; the asymptotes (dotted lines) for $S \to 0$ were calculated according to Eq. (*24*).

a number of velocity profiles, which fulfil the conditions sub *a*), have been tested. It was found, that the value of T_a calculated from Eq. (*18*), using profile (*19*), is in the best agreement with the exact solution of the referential case of a Newtonian fluid: $T_{e,Newt} = 1\cdot3120$, $T_{a,Newt} = 1\cdot309$ (difference 0·2% only). *c*) The simplicity of the relation (*19*), which is very useful in the mathematical operations, originates from the fact that a compatibility condition of the profile with the equation of motion at the point $y = 0$ (at the wall) was not *a priori* required when choosing the alternative $\mathscr{G}(Z)$-profiles. This requirement brings about complications into the form of the profiles and doesn't yield more accurate final results, analogically to other similar problems[15]. Common sense and exact solution indicate $\mathscr{G}'(0) > 1$.

The transcendent Eq. (*17*) was solved numerically using Wegstein's iteration scheme. The results of such calculations are presented in Fig 1, again using the ratio $\Delta T_a = T_a/T_{a,Newt}$ ($T_{a,Newt} = 1\cdot309$) to make a comparison of the exact and the approximate solutions easier.

RESULTS AND DISCUSSION

As is evident from Fig. 1, all the dependences obtained have a similar form: the curves are linear for small values of β, with increasing β they decrease slowly to the common asymptote $\Delta T \to 1$ for $\beta \to \infty$. This shape is similar to the relation between the effective viscosity of a PE fluid μ_{eff} and the rate of shear γ

$$\mu_{eff} = \mu + \sinh^{-1}(\gamma/C)/(\gamma B)\,. \tag{20}$$

From this relation it follows, that for $\gamma \to \infty$ $\mu_{eff} \to \mu$ and for $\gamma \to 0$ $\mu_{eff} \equiv \mu_0 \to \mu(1 + \mathrm{Ey})$: PE fluids behave as Newtonian fluids in both limiting cases, but with different viscosities. This is a well known fact, considered to be an advantage of the PE model.

The shape of the curves in Fig. 1 may be easily interpreted, after a modification of parameter β, *e.g.* in the form

$$\sqrt{(3\beta)} \equiv S = \frac{U_0/L}{C}\sqrt{\mathrm{Re}}\,. \tag{21}$$

We see, that S can be treated as a characteristic dimensionless shear rate of the analysed flow situation. The curves in Fig. 1 may be then considered to be specific forms of rheograms for classes of PE fluids, characterized by different values of Ey. Similar "pseudoreograms" resulted in the case of rotational flows of PE fluids[16].

From the analysis given above it follows that the limits for $S \to \infty$ and $S \to 0$ may be considered as two asymptotic independent cases of the flow past a rectangular wedge by two Newtonian fluids with viscosities μ and $\mu(1 + \mathrm{Ey})$. The correctness of such an interpretation can be easily demonstrated from the forms of equation (8) for the limiting cases under consideration:

$$G'''_\infty = G'^2_\infty - 2G_\infty G''_\infty - 1 \quad \text{for} \quad S \to \infty \tag{22}$$

$$G'''_0 = (G'^2_0 - 2G_0 G''_0 - 1)/(1 + \text{Ey}) \quad \text{for} \quad S \to 0 \,. \tag{23}$$

While the expression for $S \to \infty$ (Eq. (*22*)) is equivalent with the form which can be found for Newtonian fluids (*e.g.* for Ey = 0 in Eq. (*8*), or see[8]), the relation (*23*) for $S \to 0$ can be transformed into this Newtonian form after a simple modification which follows logically from what has been said previously: in this case we have to use $\mu_0 = \mu(1 + \text{Ey})$ instead of μ at the pertinent places in the derivation.

The value of the ΔT-asymptotes for $S \to 0$ can then easily be determined as

$$\lim_{S \to 0} \Delta T = (\mu_0/\mu)^{1/2} = \sqrt{(1 + \text{Ey})} \,. \tag{24}$$

As is seen from Fig. 1, the values of ΔT, calculated by both the methods for small S, approach very well the asymptotic values given by Eq. (*24*). This fact may be considered as an evidence of the inner consistency of the performed calculations, mainly as the determination of $G''(0)$ is concerned.

Considering the accuracy of the integral momentum balance method, it may be seen From Fig. 1, that in all cases this approximate method provides lower values of the shear stress at the wall when using the velocity profile (*19*). The difference between the exact and the approximate solutions are the highest at intermediate values of S, while in the limiting cases ($S \to 0$ and $S \to \infty$) the difference decreases and ΔT_a approaches the asymptotic values of 1 or $\sqrt{(1 + \text{Ey})}$, as for the exact solution. The deviation between these two solutions defined as

$$\delta_T = \frac{\Delta T_e - \Delta T_a}{\Delta T_a} \,. \, 100 \tag{25}$$

reaches its maximum values $\delta_T = 9{\cdot}2$, $11{\cdot}6$ and $13{\cdot}4\%$ for Ey = 10, 100 and 1000 respectively. From an engineering point of view such an accuracy is acceptable and comparable to that normally obtained in Newtonian hydrodynamics. This accuracy could probably be improved by using more complicated approximate velocity profiles taking into consideration the fact, that the profiles depend also on Ey and S, as it is obvious from the exact solution. Though basic recommendations for expressing these influences are missing it is clear that the calculation would be more complicated after such a modification. Hence the universality of the integral momentum balance method which is of interest for an engineer, would be lost.

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